Shorter Tours by Nicer Ears

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The $\frac{7}{5}$-approximation algorithm for TSP is presented here as follows: we start with considering instances with ear decompositions which have a very special structure. This gives rise to the first algorithm. Next, the properties of the ear decomposition are relaxed. We then sketch how an ear decomposition with the relaxed properties can be found in polytime.[1]

1 Extremely Nice Ears

Suppose we have a graph $G = (V, E)$ with an ear decomposition with the following properties (after deleting the trivial ears):

1. all short (consisting of 3 edges or less) ears are pendant (having no other ear attached to an interior node of the ear).
2. all ears are odd (have an odd number of edges)
3. the graph induced by all short ears is acyclic
4. all ears are open.

Then we can find a tour of length at most $\frac{7}{5}|V|$.

Proof. We take the best result of two algorithms. The first algorithm is the following:

1. take the edges of all short ears
2. add a minimal set of edges so that all nodes not in the interior of pending ears are connected
3. add all edges of remaining pendant ears
4. find a minimal T-join, where T is the set of nodes that have odd degree in current solution.

The number of edges in the solution for each step is:

1. $\frac{3}{2}|V_3|$, where V_3 is the set of nodes that are interior nodes of 3-ears, since 3 edges are added for every 3-ear, and each 3-ear has 2 interior nodes
2. $|V_0| - \pi_3 - 1$, where π_3 is the number of 3-ears, and V_0 is the set of nodes that is not in the interior of pending ears, since we can view this as the number of edges in a tree connecting $|V_0| - \pi_3$ nodes, after contracting the 3-ears of the first step; the fact that the graph induced by all short pendant ears is acyclic means that every contraction reduces the number of nodes by 1
3. $\leq \frac{5}{4}|V_{p,\geq 5}|$, where $V_{p,\geq 5}$ is the set of nodes that are interior nodes of pendant ears of size 5 or larger, since x edges are added for every $x - 1$ nodes in the interior, and $x \geq 5$

[1] To do: add subsection explaining how the case with closed ears can be handled, add algorithms for 2-connected subgraph problem and connected T-join problem, add section about notation and terminology in paper, which I don't follow here.

4. $\leq \frac{1}{2}|V_0|$, since only nodes in V_0 can have odd degree, and all ears are odd.

We thus get a total of at most $\frac{3}{2}|V| - \pi_3 - \frac{1}{4}|V_{p,\geq 5}| - 1 \leq \frac{3}{2}|V| - \pi - 1$ edges for algorithm 1, where π is the number of pendant ears.

The second algorithm uses the framework of Mömke and Svensson. For each non-pendant ear, create a removable pair for the ear edges incident to an interior node to which another ear is attached. For each pendant ear, make an arbitrary ear edge removable. By MS there exists a tour of at most $\frac{4}{3}|E| - \frac{2}{3}|R|$, where R denotes the set of removable edges. Note that $|E| = |V| + k - 1$, where k is the number of ears. Therefore, $\frac{4}{3}|E| - \frac{2}{3}|R| = \frac{4}{3}|V| + \frac{2}{3}\pi - \frac{4}{3}$, since for every non-pendant ear we added 2 edges to R, whilst for every pendant ear we added one edge to R.

Using the bound $\frac{4}{3}|V| + \frac{2}{3}\pi - 1$ for the second algorithm, gives a break even point at $\pi = \frac{1}{10}|V|$, and we get that the number of edges of the best of the two algorithms is bounded by $\frac{7}{5}|V| - 1$. □

2 Variations

Instead of finding a tour of length at most $\frac{7}{5}|V|$, it is sufficient to find a tour of length at most $\frac{7}{5}(LP)$ where (LP) is the optimum LP-solution for the instance.

2.1 Allowing even ears

If we do not require the second property of an extremly nice ear decomposition (all ears are odd), then the analysis of algorithm 1 changes as follows. The number of edges in the solution for each step of algorithm 1 is:

1. $2|V_2| + \frac{3}{2}|V_3| = \frac{3}{2}|V_2| + \frac{3}{2}|V_3| + \frac{1}{2}\pi_2$, where V_2 is the set of nodes that are interior nodes of 2-ears, and π_2 is the number of 2-ears
2. $|V_0| - \pi_2 - \pi_3 - 1$
3. $\leq \frac{4}{3}|V_{p,4}| + \frac{5}{4}|V_{p,\geq 5}|$
4. $\leq \frac{1}{2}(|V_0| + \varphi_0 - 1)$, where φ_0 is the number of even non pendant ears in the ear decomposition (i.e., even ears containing only nodes in V_0), by a result of Frank (1993).

We thus get a total of at most $\frac{3}{2}|V| - \frac{1}{2}\pi_2 - \pi_3 - \frac{1}{6}|V_{p,4}| - \frac{1}{4}|V_{p,\geq 5}| + \frac{1}{2}\varphi_0 - \frac{3}{2} \leq \frac{3}{2}|V| - \pi + \frac{1}{2}\pi_2 + \frac{1}{2}\pi_4 + \frac{1}{2}\varphi_0 - \frac{3}{2} \leq \frac{3}{2}|V| - \pi + \frac{1}{2}\varphi' - \frac{3}{2}$ edges for algorithm 1, where π is the number of pendant ears, and φ' is the total number of even ears in the ear decomposition.

By a result of Cheriyan, Sebő and Szigeti (2001), we also have an increased lower bound on the LP of $|V| + \varphi - 1$, where φ is the minimum number of even ears in any ear decomposition of G.

Therefore, if we relax property 2 as follows:

2. the number of even ears is minimum over all ear decompositions (i.e., equal to φ)

and we again choose to run algorithm 1 if $\pi \geq \frac{1}{10}|V|$, and algorithm 2 otherwise, we get that the number of edges is upper bounded by $\frac{7}{5}|V| + \varphi - 1$ for both algorithms, i.e. within a factor of $\frac{7}{5}$ of the LP lower bound.

2.2 Some subsets of short pendant ears form cycles

We weaken the third property, by again increasing the lower bound of the LP.

3. let E_1 be the set of edges in the short ears; E_1 is maximal in the following sense: if the number of components of (V, E_1) is $|V| - (|E_1| - \alpha)$, then $(LP) \geq |V| - 1 + \alpha$.

Note that the stronger assumption is equivalent to having $\alpha = 0$. The only change in the analysis is the number of edges in the solution for step 2 of algorithm 1, where we need an additional α number of edges. But the lower bound increases by α as well, by assumption.

By summing $\frac{1}{2}$ times the lower bounds on the LP of the previous subsection and 1 times the lower bound on the LP in this subsection, we see that both properties 2 and 3 can be relaxed simultaneously and we can still find a tour of cost at most $\frac{7}{5}(LP)$.

2.3 Allowing closed ears

The algorithm in Section 1 can be applied even if there are closed ears, and we can also define a removable pairing as before. However, in order to apply the Mömke-Svensson result (which needs the fact that there exists a probability distribution over matchings which contains each edge with probability $\frac{1}{3}$), the graph needs to be 2-vertex connected, and —even if the initial graph is 2-vertex connected— after deleting the trivial ears, this may not be the case if the ear decomposition has closed ears.

Therefore, we need to perform the algorithm block by block, where cut nodes define the boundaries between blocks. The overall result still holds, basically because of the "-1"s in the bounds.

[...]

3 Algorithm

If we can find an ear decomposition with the (relaxed) properties 1, 2 and 3 in polynomial time, then we have a $\frac{7}{5}$-approximation algorithm for the traveling salesman problem. Here are the properties for completeness' sake:

1. all short ears are pendant

2. the number of even ears is minimum over all ear decompositions (i.e., equal to φ)

3. let E_1 be the set of edges in the short ears; E_1 is maximal in the following sense: if the number of components of (V, E_1) is $|V| - (|E_1| - \alpha)$, then $(LP) \geq |V| - 1 + \alpha$.

Proposition 3.2 by Cheriyan, Sebő and Szigeti (2001) shows how to construct an open ear decomposition with property 2 in polytime. The paper shows how this ear decomposition can be modified in polytime so that property 1 holds in Section 2.

This ear decomposition can then again be modified in polytime so that property 3 holds — this is explained in Section 2 and Section 3, and can be summarized as follows: first we make sure that there are no edges between nodes in the interior of short nodes (this can be done in polytime, and is shown in Section 2). Next we solve the following matroid intersection problem: for each short ear we consider all edges incident to the interior node(s). We can choose any pair of distinct edges for an ear (if it is a 3-ear, as long as the edges are incident to different interior nodes), to be the ear (together with the edge between the interior nodes, if any) in our new ear decomposition. This (choosing a pair for each short ear) is a partition matroid. We want to maximize the number of these elements, under the condition that they do not form cycles (when contracted). This is a graphical matroid. Hence, we can solve this in polytime, by solving the matroid intersection problem.

Section 4 shows that the solution found gives the required lower bound on the LP as well, by constructing a dual solution (implicitly).

4 2-connected subgraph problem and TSP-path problem

[...]